THE FARRELL-HSIANG METHOD REVISITED

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ABSTRACT. We present a sufficient condition for groups to satisfy the Farrell-Jones Conjecture in algebraic K-theory and L-theory. The condition is formulated in terms of finite quotients of the group in question and is motivated by work of Farrell-Hsiang.

This version is different from the published version. A number of typos and an incorrect formula for the transfer before Lemma 6.3 pointed out by Holger Reich have been corrected.

INTRODUCTION

Farrell-Hsiang used in [7] a beautiful combination of controlled topology and Frobenius induction to prove that the Whitehead group of fundamental groups of compact flat Riemannian manifolds is trivial. This general method has been refined and used further by Farrell-Hsiang, Farrell-Jones and Quinn, see for example [8, 9, 11, 17]. These results belong to a much wider collection of results that ultimately led to the Farrell-Jones Conjecture [12] that predicts a formula for K- and L-theory of group rings RG. This formula describes these groups in terms of group homology and K- and L-theory of group rings RV, where V varies over the family VCyc of virtually cyclic subgroups of G. Often it is useful to consider a variant of the Conjecture where VCyc is replaced by a larger families of subgroups. For more information about the Farrell-Jones Conjecture and its applications we refer to [5, 15].

The present paper gives an axiomatic treatment of the Farrell-Hsiang method leading us to the definition of Farrell-Hsiang groups below. More generally we define a group to be a Farrell-Hsiang group with respect to a given family of subgroups $\mathcal{F}$, more or less if the Farrell-Hsiang method is applicable relative to $\mathcal{F}$. Our main result states that the Farrell-Jones Conjecture holds for these groups relative to the family $\mathcal{F}$. In the most important case $\mathcal{F}$ is the family VCyc of virtually cyclic subgroups or a family of groups for which the Farrell-Jones Conjecture relative to VCyc is known. In this case our result implies that if a group G is a Farrell-Hsiang group relative to $\mathcal{F}$, then G satisfies both the K- and L-theoretic Farrell-Jones Conjecture with coefficients in additive categories. Our main result here is used in work with Tom Farrell to prove the Farrell-Jones Conjecture for virtually poly-cylic groups [1]. We give a very brief overview of this application in an Appendix where we also discuss examples of Farrell-Hsiang groups.

In [10] Farrell-Jones used a wonderful combination of controlled topology and the dynamics of the geodesic flow on negatively curved manifolds to prove that the Whitehead group of the fundamental group of such manifolds vanishes. This Farrell-Jones method has also been refined and further used in many papers about the Farrell-Jones conjecture and the Borel conjecture, see for example [12, 13]. In [4, 2] an axiomatic treatment for this method is given that is from a formal

point of view very similar to our treatment here. In both cases a transfer and a contracting map are the main ingredients. The main difference is, that the transfer in the Farrell-Hsiang method uses a finite discrete fiber and its construction depends on Frobenius induction, whereas in the Farrell-Jones method the fiber is a compact contractible space and the transfer is essentially given by the tensor product with the singular chain complex of this fiber. Also, in applications the construction of the contracting maps is very different. In the first case subgroups of finite but large index are exploited, in the second case the dynamic of flow spaces is a key ingredient.

1. Farrell-Hsiang groups

A finite group H is said to be *hyperelementary* if it can be written as an extension $1 \to C \to H \to P \to 1$, where C is a cyclic group and P is a p-group for some prime p.

Definition 1.1 (Farrell-Hsiang group). Let $\mathcal{F}$ be a family of subgroups of the finitely generated group G. We call G a *Farrell-Hsiang group* with respect to the family $\mathcal{F}$ if the following holds for a fixed word metric d_G:

There exists a natural number N such that for every natural number n there is a surjective homomorphism $\alpha_n \colon G \to F_n$ with F_n a finite group such that the following condition is satisfied. For any hyperelementary subgroup H of F_n we set $\overline{H} := \alpha_n^{-1}(H)$ and require that there exists a simplicial complex E_H of dimension at most N with a cell preserving simplicial $\overline{H}$-action whose stabilizers belong to $\mathcal{F}$, and an $\overline{H}$-equivariant map $f_H \colon G \to E_H$ such that $d_G(g, h) < n$ implies $d^1_{E_H}(f_H(g), f_H(h)) < \frac{1}{n}$ for all $g, h \in G$, where $d^1_{E_H}$ is the l^1-metric on E_H.

Theorem 1.2 (Main Theorem). *Let G be a Farrell-Hsiang group with respect to the family $\mathcal{F}$ in the sense of Definition 1.1. Then G satisfies the K-theoretic and L-theoretic Farrell-Jones Conjecture with additive categories as coefficients with respect to the family $\mathcal{F}$.*

For the precise formulation and discussion of the Farrell-Jones Conjecture with coefficients in additive categories we refer to [3].

Remark 1.3. Definition 1.1 can be weakened if one is only interested in the L-theoretic Farrell-Jones conjecture. In this case it suffices to consider all subgroups H of F that are either 2-hyperelementary or p-elementary for some prime $p \neq 2$. In other words p-hyperelementary subgroups that are not p-elementary can be ignored for all odd primes p.

2. Categorical preliminaries

2.A. **Additive G-categories with involutions.** In this paper we will understand notions like additive category (with involution) or additive G-category (with involution) always in the strict sense. This means that all our additive categories will come with a strictly associative functorial direct sum $(M, N) \mapsto M \oplus N$ and an involution I on an additive category $\mathcal{B}$ is a contravariant functor $I \colon \mathcal{B} \to \mathcal{B}$ with $I^2 = \mathrm{id}_\mathcal{B}$. When we talk about an additive G-category, the (right) G-action is understood to be in the strict sense, i.e., for every $g \in G$ we have a functor $R_g \colon \mathcal{B} \to \mathcal{B}$ of additive categories such that $R_h \circ R_g = R_{gh}$ for $g, h \in G$. If $\mathcal{B}$ comes with an involution $I_\mathcal{B}$, then we require $I_\mathcal{B} \circ R_g = R_g \circ I_\mathcal{B}$ for all $g \in G$.

Remark 2.1. Often a more general definition of additive categories with involutions is used, where the equality $I^2 = \mathrm{id}_{\mathcal{B}}$ is replaced by a natural equivalence $E \colon I^2 \to \mathrm{id}_{\mathcal{B}}$. One may also consider additive categories with weak G-actions. We refer to [3], where all these notion are explained, it is shown how one can replace the weak versions by equivalent strict versions, and – most important – that for a proof of the Farrell-Jones Conjecture it suffices to consider the strict versions (see [3, Theorem 0.2]). We use the strict versions to simplify some formulas. The only slight disadvantage of this is, that it forces us to replace some very natural categories by some slightly less natural categories, see for instance the definition of $\mathtt{mod}_{\mathbb{Z}}$ below.

A functor between additive categories with involutions $(\mathcal{B}, I)$ and $(\mathcal{B}', I')$ is a pair (F, E) where $F \colon \mathcal{B} \to \mathcal{B}'$ is an additive functor, and $E \colon F \circ I \to I' \circ F$ is a natural equivalence such that $I'(E(M)) = E(I(M))$ for all objects $M \in \mathcal{B}$. If $F \circ I = I' \circ F$ and $E = \mathrm{id}$, then the functor is said to be strict. Most of our functors will be strict, but not all of them. Functors between additive categories with involutions induce maps in L-theory.

2.B. The category $\mathtt{mod}_{\mathbb{Z}}$ of based finitely generated free abelian groups.

On the category of finitely generated free abelian groups the involution $T \mapsto T^* := \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ is not strict since T is not $(T^*)^*$ on the nose. To fix this inconvenience we will consider the following additive category with involution $\mathtt{mod}_{\mathbb{Z}}$ instead. The objects of $\mathtt{mod}_{\mathbb{Z}}$ are $\mathbb{Z}^n$, $n = 0, 1, 2, \ldots$. The set of morphisms $\mathrm{mor}_{\mathtt{mod}_{\mathbb{Z}}}(\mathbb{Z}^n, \mathbb{Z}^m)$ is given by $n \times m$-matrices. Composition is given by the usual matrix multiplication. The direct sum is given by $\mathbb{Z}^n \oplus \mathbb{Z}^m = \mathbb{Z}^{n+m}$. The involution on $\mathtt{mod}_{\mathbb{Z}}$ acts as the identity on objects and as transposition of matrices on morphisms. For an additive category $\mathcal{A}$ there is a functor

$$- \otimes_{\mathbb{Z}} - \colon \mathtt{mod}_{\mathbb{Z}} \times \mathcal{A} \to \mathcal{A}$$

defined by $\mathbb{Z}^n \otimes_{\mathbb{Z}} M = \bigoplus_{i=1}^n M$, see for example [2, Section 6]. This functor is bilinear on morphisms groups. It follows that given an object $\mathbb{Z}^n$ in $\mathtt{mod}_{\mathbb{Z}}$, the functor $\mathbb{Z}^n \otimes_{\mathbb{Z}} - \colon \mathcal{A} \to \mathcal{A}$ is a functor of additive categories, and given an object $M \in \mathcal{A}$, the functor $- \otimes_{\mathbb{Z}} M \colon \mathtt{mod}_{\mathbb{Z}} \to \mathcal{A}$ is a functor of additive categories. If $\mathcal{A}$ comes with an involution, then $\mathtt{mod}_{\mathbb{Z}} \times \mathcal{A}$ inherits the obvious product involution and $- \otimes_{\mathbb{Z}} -$ is compatible with the involutions.

2.C. The category $\mathtt{mod}_{(\mathbb{Z},G)}$ of $\mathbb{Z}G$-modules which are finitely generated free as abelian groups.

Let G be a group. We define the following additive category with involution $\mathtt{mod}_{(\mathbb{Z},G)}$. Objects in $\mathtt{mod}_{(\mathbb{Z},G)}$ are pairs $(\mathbb{Z}^n, \rho)$ where $\rho \colon G \to \mathrm{GL}(n, \mathbb{Z})$ is a group homomorphism. A morphism $f \colon (\mathbb{Z}^n, \rho) \to (\mathbb{Z}^m, \eta)$ is a morphism $f \colon \mathbb{Z}^n \to \mathbb{Z}^m$ in $\mathtt{mod}_{\mathbb{Z}}$ which is compatible with the homomorphisms ρ and η, i.e., $\eta(g) \circ f = f \circ \rho(g)$ for all $g \in G$. The direct sum is given by the direct sum in $\mathtt{mod}_{\mathbb{Z}}$. Define an involution $I_{\mathtt{mod}_{(\mathbb{Z},G)}}$ on $\mathtt{mod}_{(\mathbb{Z},G)}$ as follows. It sends an object $(\mathbb{Z}^n, \rho)$ to the object $(\mathbb{Z}^n, \rho^*)$, where $\rho^*(g)$ is defined by $I_{\mathtt{mod}_{\mathbb{Z}}}(\rho(g^{-1}))$. A morphism $f \colon (\mathbb{Z}^n, \rho) \to (\mathbb{Z}^m, \eta)$ is sent to the morphism given by $I_{\mathtt{mod}_{\mathbb{Z}}}(f)$.

Of course $\mathtt{mod}_{(\mathbb{Z},G)}$ is a model for the category of $\mathbb{Z}G$-modules which are finitely generated free as abelian groups and has the extra feature that the involution is strict.

Let $\alpha \colon H \to G$ be a group homomorphism. We obtain a functor of additive categories with involution called *restriction*

$$\mathrm{res}_\alpha \colon \mathtt{mod}_{(\mathbb{Z},G)} \to \mathtt{mod}_{(\mathbb{Z},H)},$$

which sends an object $(\mathbb{Z}^n, \rho)$ to the object $(\mathbb{Z}^n, \rho \circ \alpha)$ and a morphism $f \colon (\mathbb{Z}^n, \rho) \to (\mathbb{Z}^m, \eta)$ to the morphism $f \colon (\mathbb{Z}^n, \rho \circ \alpha) \to (\mathbb{Z}^m, \eta \circ \alpha)$.

Next we define the induction functor for a subgroup H of G of finite index

$$\operatorname{ind}_H^G \colon \operatorname{mod}_{(\mathbb{Z},H)} \to \operatorname{mod}_{(\mathbb{Z},G)}.$$

It will depend on a choice of representatives $g_0, \ldots, g_{m-1} \in G$ for $G/H = \{g_0\overline{H}, \ldots, g_{m-1}\overline{H}\}$. This choice will not matter in the sequel, since for two such choices we obtain a unique natural equivalence of the corresponding functors of additive categories with involution. Consider an object $(\mathbb{Z}^n, \rho)$ in $\operatorname{mod}_{(\mathbb{Z},H)}$. The image under ind_H^G is the object $(\mathbb{Z}^{m\cdot n}, \eta)$, where $\eta(g) \in \operatorname{GL}(m \cdot n, \mathbb{Z})$ for $g \in G$ is the morphism in $\operatorname{mod}_{\mathbb{Z}}$ given by the matrix whose entry at $(kn + i, k'n + i')$ is 0 if $gg_{k'}H \neq g_k H$, and is $\rho\big(g_k^{-1}gg_{k'}\big)_{i,i'}$ if $gg_{k'}H = g_k H$. Here $0 \leq k, k' \leq m - 1$, $1 \leq i, i' \leq n$.

Remark 2.2. Our above definition of ind_H^G may appear unnatural. But the only reason for this is our choice of the category $\operatorname{mod}_{(\mathbb{Z},H)}$; it really is the usual definition of induction:

Let $(\mathbb{Z}^n, \rho)$ be an object of $\operatorname{mod}_{(\mathbb{Z},H)}$. Then $\mathbb{Z}^n$ becomes an $\mathbb{Z}[H]$-module via ρ. We have the following isomorphism of $\mathbb{Z}$-modules

$$\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}\mathbb{Z}^n \cong \bigoplus_{j=0}^{m-1} \mathbb{Z}[g_j H]\otimes_{\mathbb{Z}[H]}\mathbb{Z}^n \cong \bigoplus_{j=0}^{m-1} \mathbb{Z}^n \cong \mathbb{Z}^{m\cdot n}$$

and the above formula for η describes how the action of G on $\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}\mathbb{Z}^n$ conjugates to an action on $\mathbb{Z}^{m\cdot n}$ under the above isomorphism.

3. THE OBSTRUCTION CATEGORY $\mathcal{O}^G(E, Z, d; \mathcal{A})$

Let E be a G-space and (Z, d) be a quasi-metric space with a free, proper and isometric G-action. In this section we will review the the additive category $\mathcal{O}^G(E, Z, d; \mathcal{A})$ that was originally defined in [4, Section 3], see also [2, Section 4]. If $\mathcal{A}$ is an additive category with involution, then $\mathcal{O}^G(E, Z, d; \mathcal{A})$ is an additive category with involution.

3.A. **Objects.** Objects in $\mathcal{O}^G(E, Z, d; \mathcal{A})$ are given by sequences $M = (M_y)_{y \in Z \times E \times [1,\infty)}$ of objects from $\mathcal{A}$ subject to the following conditions.

    (i) *G-compact support over $Z \times E$.* There is a compact subset K of $Z \times E$ such that $M_{z,e,t} = 0$ whenever $(z, e) \notin G \cdot K$.

    (ii) *Locally finiteness.* For all $y \in Z \times E \times [1, \infty)$ there exists an open neighborhood U such that $\{y \in U \mid M_y \neq 0\}$ is finite.

    (iii) *G-equivariance.* For all $y \in Z \times E \times [1, \infty)$ and $g \in G$ we have $M_{gy} = g(M_y)$. Here $gy = (gz, ge, t)$ for $y = (z, e, t)$.

The involution $I_{\mathcal{O}}$ on $\mathcal{O}^G(E, Z, d_G; \mathcal{A})$ acts on objects point-wise, i.e., we have $(I_{\mathcal{O}}(M))_{z,e,t} = I_{\mathcal{A}}(M_{z,e,t})$.

3.B. **Morphisms.** Let $M = (M_y)_{y \in Z \times E \times [1,\infty)}$, $N = (M_y)_{y \in Z \times E \times [1,\infty)}$ be objects from $\mathcal{O}^G(E, Z, d)$. A morphism $\psi \colon M \to N$ in $\mathcal{O}^G(E, Z, d)$ is given by a sequence $\psi = (\psi_{y,y'})_{y,y' \in Z \times E \times [1,\infty)}$ of morphisms $\psi_{y,y'} \colon M_{y'} \to N_y$ in $\mathcal{A}$ subject to the following conditions.

    (i) *Row and column finiteness.* For all $y \in Z \times E \times [1, \infty)$ the set $\{y' \mid \psi_{y,y'} \neq 0 \text{ or } \psi_{y',y} \neq 0\}$ is finite.

    (ii) *Metric control over Z.* There is $R > 0$ (depending on ψ) such that $\psi_{y,y'} = 0$ whenever $y = (z, e, t)$, $y' = (z', e', t')$ with $d(z, z') > R$.

    (iii) *Metric control over $[1, \infty)$.* There is $A > 0$ (depending on ψ) such that $\psi_{y,y'} = 0$ whenever $y = (z, e, t)$, $y' = (z', e', t')$ with $|t - t'| > A$.

(iv) *G-continuous control over $E\times[1,\infty)$.* Let $e_0 \in E$, V be an G_{e_0}-invariant neighborhood of e_0 and $b > 0$. (Here $G_{e_0} = \{g \mid ge_0 = e_0\}$.) Then we require the existence of $B > 0$ and a G_{e_0}-invariant neighborhood U of e_0 such that $\psi_{y,y'} = \psi_{y',y} = 0$ whenever $y = (z,e,t)$, $y' = (z',e',t')$ with $(e,t) \in U\times(B,\infty)$ and $(e',t') \notin V\times(b,\infty)$.

(v) *G-equivariance.* For all $y, y' \in Z\times E\times[1,\infty)$ and $g \in G$ we have $\psi_{gy,gy'} = g(\psi_{y,y'})$.

For the constructions in this paper the second condition will be the most important condition and we will say that ψ is *R-controlled* if it is satisfied for a given $R > 0$. Addition and composition of morphisms is defined as for matrices: $(\psi + \psi')_{y,y'} = \psi_y + \psi'_{y'}$ and $(\psi \circ \psi')_{y,y''} = \sum_{y'} \psi_{y,y'} \circ \psi'_{y',y''}$. The involution is on morphisms defined by the formula $(I_{\mathcal{O}}(\psi))_{y,y'} = I_{\mathcal{A}}(\psi_{y',y})$.

We will often drop $\mathcal{A}$ from the notation and write $\mathcal{O}^G(E, Z, d)$ instead of $\mathcal{O}^G(E, Z, d; \mathcal{A})$.

3.C. **Functoriality.** In this paper we will only need the functoriality of $\mathcal{O}^G(E, Z, d; \mathcal{A})$ in the Z-variable. Let (Z, d) and (Z', d') be quasi-metric spaces with free, proper and isometric G-actions. Let $f\colon Z \to Z'$ be G-equivariant continuous map such that for any $r > 0$ there is $R > 0$ such that $d'(f(z_0), f(z_1)) < R$ whenever $d(z_0, z_1) < r$. Then f induces a functor $f_*\colon \mathcal{O}^G(E, Z, d) \to \mathcal{O}^G(E, Z', d')$ which is given by $(f_*(M))_{e,z,t} = \bigoplus_{z' \in f^{-1}(z)} M_{e,z',t}$. (The condition ensures that metric control over Z is turned into metric control over Z'; the G-compact support condition for objects ensures that the sum in the definition of f_* is finite.) Strictly speaking f_* is only defined up to natural equivalence because the direct sum may only be defined up to canonical isomorphism. (Our assumptions on $\mathcal{A}$ only imply that sums over *ordered* finite index set are canonically defined.)

3.D. $\mathcal{O}^G(E, G, d)$ **as the obstruction to the Farrell-Jones conjecture.** The following result is a consequence of [2, Theorem 5.2].

Theorem 3.1. *Let G be a finitely generated group, d_G a word metric on G and $\mathcal{F}$ be a family of subgroups.*

(i) *Assume that $K_*(\mathcal{O}^G(E_{\mathcal{F}}G, G, d_G))$ is trivial in all degrees. Then the K-theory assembly map $H^G_*(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \to K_*(\int_G \mathcal{A})$ is an isomorphism.*

(ii) *Assume that $L_*(\mathcal{O}^G(E_{\mathcal{F}}G, G, d_G))$ is trivial in all degrees. Then the L-theory assembly map $H^G_*(E_{\mathcal{F}}G; \mathbf{L}_{\mathcal{A}}) \to L_*(\int_G \mathcal{A})$ is an isomorphism.*

3.E. **The controlled product category.** Let (Z_n, d_n) be a sequence of quasi-metric spaces with free, proper and isometric G-actions. Consider the product category $\prod_{n\in\mathbb{N}} \mathcal{O}^G(E, Z_n, d_n)$. A morphism $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be R-controlled for $R > 0$ is φ_n is R-controlled for all n. We define $\mathcal{O}^G(E, (Z_n, d_n)_{n\in\mathbb{N}})$ as the category whose objects are objects from the product category and whose morphisms are morphisms from the product category that are R-controlled for some R. There is for any k a canonical projection functor $\mathcal{O}^G(E, (Z_n, d_n)_{n\in\mathbb{N}}) \to \mathcal{O}^G(E, Z_k, d_k)$.

4. THE CORE OF THE PROOF OF THE MAIN THEOREM 1.2

Let G be a Farrell-Hsiang group with respect to $\mathcal{F}$. Let N be the number appearing in Definition 1.1. For $n \in \mathbb{N}$ there is then $\alpha_n\colon G \to F_n$, a surjective group homomorphism onto a finite group F_n, such that the following holds: For any hyperelementary subgroup H of F_n and $\overline{H} := \alpha_n^{-1}(H)$ there is a simplicial complex E_H of dimension at most N with a cell preserving simplicial $\overline{H}$-action whose stabilizers belong to $\mathcal{F}$, and an $\overline{H}$-equivariant map $f_H\colon G \to E_H$ such that $d_G(g, h) < n$ implies $d^1_{E_H}(f_H(g), f_H(h)) < \frac{1}{n}$ for all $g, h \in G$, where $d^1_{E_H}$ is the l^1-metric on E.

Here we write $\overline{H}$ for $\alpha_n^{-1}(H)$ and we will use this convention throughout the remainder of this paper. We denote by $\mathcal{H}_n$ the family of hyperelementary subgroups of F_n. We set $X_n := G \times \coprod_{H \in \mathcal{H}_n} \mathrm{ind}_{\overline{H}}^G E_H$ and $S_n := G \times \coprod_{H \in \mathcal{H}_n} G/\overline{H}$. We equip X_n and S_n with diagonal G-action. We will use the quasi-metrics d_{X_n} on X_n and d_{S_n} on S_n defined by

$$d_{X_n}((g,x),(h,y)) \quad := \quad d_G(g,h) + n \cdot d^1_{\mathrm{ind}_{\overline{H}}^G E_H}(x,y),$$

$$d_{S_n}((g,a\overline{H}),(h,b\overline{K})) \quad := \quad \begin{cases} d_G(g,h) & \text{if } K = H \text{ and } a\overline{H} = b\overline{K}, \\ \infty & \text{otherwise.} \end{cases}$$

Here $g,h,a,b \in G$, $x,y \in X_n$, $H,K \in \mathcal{H}_n$ and $d^1_{\mathrm{ind}_{\overline{H}}^G E_H}$ is the l^1-metric on $\mathrm{ind}_{\overline{H}}^G E_H$. Abbreviate $E := E_{\mathcal{F}}G$. The proof of Theorem 1.2 is organized around the following diagram of additive categories and functors.

(4.1)
$$\bigoplus_{n \in \mathbb{N}} \mathcal{O}^G(E, X_n, d_{X_n})$$

$$\downarrow I$$

$$\mathcal{O}^G(E, (S_n, d_{S_n})_{n\in\mathbb{N}}) \xrightarrow{\quad F \quad} \mathcal{O}^G(E, (X_n, d_{X_n})_{n\in\mathbb{N}})$$

$$\downarrow P_k \qquad\qquad\qquad\qquad\qquad \downarrow Q_k$$

$$\mathcal{O}^G(E, G, d_G) \xrightarrow{\quad \mathrm{id} \quad} \mathcal{O}^G(E, G, d_G)$$

Explanations follow. The functors P_k and Q_k are defined as compositions

$$\mathcal{O}^G(E, (S_n, d_{S_n})_{n\in\mathbb{N}}) \to \quad \mathcal{O}^G(E, S_k, d_{S_k}) \to \quad \mathcal{O}^G(E, G, d_G)$$
$$\mathcal{O}^G(E, (X_n, d_{X_n})_{n\in\mathbb{N}}) \to \quad \mathcal{O}^G(E, X_k, d_{X_k}) \to \quad \mathcal{O}^G(E, G, d_G)$$

where in both cases the first functor is the projection on the k-th factor, and the second functor is induced by the canonical projection $p_k \colon S_k = G \times \coprod_{H \in \mathcal{H}_k} G/\overline{H} \to G$ and $q_k \colon X_k = G \times \coprod_{H \in \mathcal{H}_k} \mathrm{ind}_{\overline{H}}^G E_H \to G$ respectively. The functor I is the canonical inclusion. The functor F will be constructed in Proposition 7.1. We have the following facts.

   (i) For all $a \in K_*(\mathcal{O}^G(E, G, d_G))$ and $b \in L_*(\mathcal{O}^G(E, G, d_G))$ there are $\hat{a} \in K_*(\mathcal{O}^G(E, (S_n, d_{S_n})_{n\in\mathbb{N}}))$ and $\hat{b} \in L_*(\mathcal{O}^G(E, (S_n, d_{S_n})_{n\in\mathbb{N}}))$ such that for all k we have $(K_n(P_k))(\hat{a}) = a$ and $(L_*(Q_k))(\hat{b}) = b$. This will be proved in Theorem 6.5.
   (ii) For all k we have $Q_k \circ F = P_k$, see Proposition 7.1.
  (iii) The functor I induces an isomorphism in K- and L-theory. For K-theory this follows from [4, Theorem 7.2]. This result only depends on the properties of K-theory listed in [2, Theorem 5.1]. Since these properties are also enjoyed by L-theory, I induces an isomorphism in L-theory as well.

Proof of Theorem 1.2. Because of Theorem 3.1 it suffices to show that the K- and L-theory of $\mathcal{O}^G(E, G, d_G)$ is trivial. Let $a \in K_n(\mathcal{O}^G(E, G, d_G))$ and $b \in L_n(\mathcal{O}^G(E, G, d_G))$. By the first fact there are $\hat{a} \in K_n(\mathcal{O}^G(E, (S_n, d_{S_n})_{n\in\mathbb{N}}))$ and $\hat{b} \in L_n(\mathcal{O}^G(E, (S_n, d_{S_n})_{n\in\mathbb{N}}))$ such that for all k we have $(K_n(P_k))(\hat{a}) = a$ and $(L_n(P_k))(\hat{b}) = b$. It is a consequence of the third fact that for sufficient large k we have $(K_n(Q_k \circ F))(\hat{a}) = 0$ and $(L_n^{-\infty}(Q_k \circ F))(\hat{b}) = 0$. Using the second fact we conclude $a = (K_n(P_k))(\hat{a}) = (K_n(Q_k) \circ F)(\hat{a}) = 0$ and $b = (L_n(P_k))(\hat{b}) = (L_n(Q_k \circ F))(\hat{b}) = 0$. (Compare [4, p.45, Proof of Theorem 1.1].) $\qquad\square$

5. ABSTRACT TRANSFERS FOR ADDITIVE CATEGORIES

5.A. **Swan group and Dress' equivariant Witt group.** We have introduced the additive category with involutions $\mathrm{mod}_{(\mathbb{Z},G)}$ in Section 2. Recall that it is equivalent to the category of $\mathbb{Z}G$-modules which are finitely generated free as $\mathbb{Z}$-modules. We will use the exact structure on $\mathrm{mod}_{(\mathbb{Z},G)}$ where a sequence is called exact if it is exact as a sequence of $\mathbb{Z}[G]$-modules (or equivalently as a sequence of abelian groups). Notice that with this exact structure not all exact sequences are split exact over $\mathbb{Z}G$. The *Swan group* and *Dress' equivariant Witt group* are defined with respect to this exact structure as corresponding Grothendieck or Witt groups

$$\mathrm{Sw}(\mathbb{Z}, G) := G_0(\mathrm{mod}_{(\mathbb{Z},G)}) \quad \text{and} \quad \mathrm{GW}(\mathbb{Z}, G) := W(\mathrm{mod}_{(\mathbb{Z},G)}),$$

see [18, 6, 14]. Both of these become rings via the tensor product over $\mathbb{Z}$, equipped with the diagonal G-action, but we will not need this ring structure and ignore it in this paper. By $1_{\mathrm{Sw}} \in \mathrm{Sw}(\mathbb{Z}, G)$ we will denote the class of $\mathbb{Z}$ with the trivial G-action and analogously for $1_{\mathrm{GW}} \in \mathrm{GW}(\mathbb{Z}, G)$. (These are of course the units for the ring structures.)

For a group homomorphism $\alpha\colon H \to G$ there are restriction maps

$$\mathrm{res}_\alpha\colon \mathrm{Sw}(\mathbb{Z}, G) \to \mathrm{Sw}(\mathbb{Z}, H)$$
$$\mathrm{res}_\alpha\colon \mathrm{GW}(\mathbb{Z}, G) \to \mathrm{GW}(\mathbb{Z}, H)$$

coming from the restriction functor $\mathrm{res}_\alpha\colon \mathrm{mod}_{(\mathbb{Z},G)} \to \mathrm{mod}_{(\mathbb{Z},H)}$. Clearly, we have $\mathrm{res}_\alpha(1_{\mathrm{Sw}}) = 1_{\mathrm{Sw}}$ and $\mathrm{res}_\alpha(1_{\mathrm{GW}}) = 1_{\mathrm{GW}}$.

For a subgroup $H \subseteq G$ of finite index there are induction homomorphisms

$$\mathrm{ind}_H^G\colon \mathrm{Sw}(\mathbb{Z}, H) \to \mathrm{Sw}(\mathbb{Z}, G)$$
$$\mathrm{ind}_H^G\colon \mathrm{GW}(\mathbb{Z}, H) \to \mathrm{GW}(\mathbb{Z}, G)$$

coming from the induction functor $\mathrm{ind}_H^G\colon \mathrm{mod}_{(\mathbb{Z},H)} \to \mathrm{mod}_{(\mathbb{Z},G)}$.

Actually, both $\mathrm{Sw}(\mathbb{Z}, -)$ and $\mathrm{GW}(\mathbb{Z}, -)$ are Green functors. Later on we will make crucial use of the following results due to Swan and Dress.

Theorem 5.1 (Swan [18];Dress[6]). *Let F be a finite group. Let $\mathcal{H}$ be the family of hyperelementary subgroups of F.*

(i) There are $\tau_H \in \mathrm{Sw}(\mathbb{Z}, H)$, $H \in \mathcal{H}$ such that

$$1_{\mathrm{Sw}} = \sum_{H \in \mathcal{H}} \mathrm{ind}_H^F(\tau_H) \in \mathrm{Sw}(\mathbb{Z}, F).$$

(ii) There are $\sigma_H \in \mathrm{GW}(\mathbb{Z}, H)$, $H \in \mathcal{H}$ such that

$$1_{\mathrm{GW}} = \sum_{H \in \mathcal{H}} \mathrm{ind}_H^F(\sigma_H) \in \mathrm{GW}(\mathbb{Z}, F).$$

Remark 5.2. In Theorem 5.1 (ii) the family $\mathcal{H}$ can be replaced by the family of subgroups H of F that are either 2-elementary or p-hyperelementary for some prime $p \neq 2$.

5.B. **Action of $\mathrm{Sw}(\mathbb{Z}, G)$ in K-theory.** Let R be a ring and G be a group. Denote by $\mathrm{mod}_{R[G]}$ the category of finitely generated projective $R[G]$-modules. The tensor product over $\mathbb{Z}$, equipped with the diagonal G-action, $(T, M) \mapsto T \otimes_{\mathbb{Z}}^{\Delta} M$ defines a bilinear functor

$$-\otimes_{\mathbb{Z}}^{\Delta}-\colon \mathrm{mod}_{(\mathbb{Z},G)} \times \mathrm{mod}_{R[G]} \to \mathrm{mod}_{R[G]}.$$

In particular, we obtain a functor $T \otimes_{\mathbb{Z}}^{\Delta} -\colon \mathrm{mod}_{R[G]} \to \mathrm{mod}_{R[G]}$ for every module $T \in \mathrm{mod}_{(\mathbb{Z},G)}$. Applying K-theory we obtain a endomorphism $K_*(T \otimes_{\mathbb{Z}}^{\Delta} -)$ of

$K_*(R[G])$, This endomorphism depends only on the class of T in $\mathrm{Sw}(\mathbb{Z}, G)$ and yields a pairing

$$\mu\colon \mathrm{Sw}(\mathbb{Z}, G) \otimes K_*(R[G]) \to K_*(R[G])$$

such that $\mu([T] \otimes a) = K_*(T \otimes_{\mathbb{Z}}^{\Delta} -)(a)$ for all $a \in K_*(R[G])$ (see [16, Corollary 1 on page 106]). This has a generalization as follows. For an additive category $\mathcal{B}$ a functor

$$F\colon \mathrm{mod}_{(\mathbb{Z}, G)} \times \mathcal{B} \to \mathcal{B}$$

is said to be *exact* if F is bilinear and for any short exact sequence (which is *not* necessarily split exact) $0 \to S_0 \xrightarrow{i} S_1 \xrightarrow{p} S_2 \to 0$ in $\mathrm{mod}_{(\mathbb{Z}, G)}$ and any object B in $\mathcal{B}$ the induced sequence $0 \to F(S_0, B) \xrightarrow{F(i, \mathrm{id}_B)} F(S_1, B) \xrightarrow{F(p, \mathrm{id}_B)} F(S_2, B) \to 0$ in $\mathrm{mod}_{(\mathbb{Z}, G)}$ is exact in $\mathcal{B}$. Recall that a sequence $0 \to B_0 \xrightarrow{j} B_1 \xrightarrow{q} B_2 \to 0$ in an additive category $\mathcal{B}$ is called exact if it is split exact, i.e, $q \circ j = 0$ and there exists a morphism $s\colon B_2 \to B_0$ such that $q \circ s = \mathrm{id}_{B_2}$ and $j \oplus s\colon B_0 \oplus B_2 \to B_1$ is an isomorphism.

Proposition 5.3. *Given an exact functor $F\colon \mathrm{mod}_{(\mathbb{Z}, G)} \times \mathcal{B} \to \mathcal{B}$, there is a bilinear pairing*

$$\mu_F\colon \mathrm{Sw}(\mathbb{Z}, G) \otimes K_*(\mathcal{B}) \to K_*(\mathcal{B})$$

such that $\mu_F([T] \otimes a) = K_*(F(T, -))(a)$ for all $a \in K_*(\mathcal{B})$.

5.C. **Action of $\mathrm{GW}(\mathbb{Z}, G)$ in L-theory.** Let $\mathcal{B}$ be an additive category with a strict involution $I_{\mathcal{B}}$ and

$$F\colon \mathrm{mod}_{(R, G)} \times \mathcal{B} \to \mathcal{B}$$

be an exact functor which is compatible with the involutions, i.e., $I_{\mathcal{B}}(F(-, -)) = F(-^*, I_{\mathcal{B}}(-))$. Then for a module $T \in \mathrm{mod}_{(G, \mathbb{Z})}$ the linear functor $F(T, -)\colon \mathcal{B} \to \mathcal{B}$ does a priori not induce a map in L-theory because no canonical isomorphism $I_{\mathcal{B}}(F(T, M)) \to F(T, I_{\mathcal{B}}(M))$ is provided. To fix this, we pick an isomorphism $\varphi\colon T \to T^*$ in $\mathrm{mod}_{(\mathbb{Z}, G)}$ such that $\varphi^* = \varphi$, so (T, φ) is a symmetric form in $\mathrm{mod}_{(\mathbb{Z}, G)}$. Then

$$F(\varphi, \mathrm{id}_{I_{\mathcal{B}}(-)})\colon F(T, I_{\mathcal{B}}(-)) \to F(T^*, I_{\mathcal{B}}(-)) = I_{\mathcal{B}}(F(T, -))$$

is a natural isomorphism and $F((T, \varphi), -) := (F(T, -), F(\varphi, \mathrm{id}_{I_{\mathcal{B}}(-)}))\colon \mathcal{B} \to \mathcal{B}$ is a functor of additive categories with involutions. There is the following analog of Proposition 5.3.

Proposition 5.4. *Given an exact functor $F\colon \mathrm{mod}_{(G, \mathbb{Z})} \times \mathcal{B} \to \mathcal{B}$ that is compatible with involutions, there is a bilinear pairing*

$$\mu_F\colon \mathrm{GW}(\mathbb{Z}, G) \otimes L_*(\mathcal{B}) \to L_*(\mathcal{B})$$

such that $\mu_F([T, \varphi] \otimes b) = L_*(F((T, \varphi), -))(b)$ for all $b \in L_*(\mathcal{B})$ and all symmetric forms (T, φ) over $\mathrm{mod}_{(\mathbb{Z}, G)}$.

Proof. If $\mathcal{B}$ is the category of finitely generated free $R[G]$-modules and F is the diagonal tensor product, then this is worked out in detail in [6] and [14]. The case of general F and $\mathcal{B}$ is not more complicated. $\qquad\square$

6. The transfer

6.A. **Transfer functors.** Let G be a group with a metric d_G and E be a G-space. We define a functor

$$\mathrm{tr}\colon \mathrm{mod}_{(\mathbb{Z}, G)} \times \mathcal{O}^G(E, G, d_G) \to \mathcal{O}^G(E, G, d_G)$$

as follows. Recall that we have a tensor product functor $\mathrm{mod}_{\mathbb{Z}} \times \mathcal{A} \to \mathcal{A}$ which is compatible with the involution on $\mathrm{mod}_{\mathbb{Z}}$ and $\mathcal{A}$, see Section 2. For objects

$T = (\mathbb{Z}^n, \rho) \in \text{mod}_{(\mathbb{Z},G)}$ and $M = (M_z)_{z \in G \times E \times [1,\infty)} \in \mathcal{O}^G(E, G, d_G)$ we define $\text{tr}(T, M) \in \mathcal{O}^G(E, G, d_G)$ by setting

$$\big(\text{tr}(T, M)\big)_z := \mathbb{Z}^n \otimes_{\mathbb{Z}} M_z$$

for $z \in G \times E \times [1, \infty)$. For morphisms $f \in \text{mod}_{(\mathbb{Z},G)}$ and $\psi = (\psi_{z,z'})_{z,z' \in G \times E \times [1,\infty)} \in \mathcal{O}^G(E, G, d_G)$ we define $\text{tr}(f, \psi)$ by setting

$$\big(\text{tr}(f, \psi)\big)_{z,z'} := (f \circ \rho(g^{-1}g')) \otimes_{\mathbb{Z}} \psi_{z,z'}$$

for $z = (g, e, t), z' = (g', e', t') \in G \times E \times [1, \infty)$.

Lemma 6.1. *The functor* tr *is exact. It is compatible with involutions if* $\mathcal{A}$ *comes with a (strict) involution.*

Proof. The compatibility with involutions follows from the same compatibility for $\otimes_{\mathbb{Z}}$.

Consider an exact sequence $0 \to S_0 \xrightarrow{i} S_1 \xrightarrow{q} S_2 \to 0$ in $\text{mod}_{(\mathbb{Z},G)}$. We have to show that for any object M in $\mathcal{O}^G(E, G, d_G)$ that the composite $\text{tr}(q, \text{id}_M) \circ \text{tr}(i, \text{id}_M)$ is trivial, $\text{tr}(q, \text{id}_M) \colon \text{tr}(S_1, M) \to \text{tr}(S_2, M)$ is split surjective, and that the direct sum of the splitting and the map $\text{tr}(i, \text{id}_M)$ yields an isomorphism $\text{tr}(S_0, M) \oplus \text{tr}(S_2, M) \xrightarrow{\cong} \text{tr}(S_1, M))$. We only construct the splitting of $\text{tr}(q, \text{id}_M)$. Let $s \colon S \hookrightarrow T$ be a section for q as a map of $\mathbb{Z}$-modules. Then a section $\hat{s}$ for $\text{tr}(q, \text{id}_M)$ is defined by setting

$$(\hat{s})_{z,z'} := \begin{cases} s \otimes \text{id}_{M_z} & \text{if } z = z' \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Remark 6.2. To illustrate the proof above consider an epimorphism $p \colon M \to N$ of $\mathbb{Z}G$-modules which are finitely generated free as abelian groups and the induced map of $\mathbb{Z}G$-modules (with respect to the diagonal action) $p \otimes_{\mathbb{Z}} \text{id}_{\mathbb{Z}G} \colon M \otimes_{\mathbb{Z}} \mathbb{Z}G \to N \otimes_{\mathbb{Z}} \mathbb{Z}G$. We want to construct a $\mathbb{Z}G$-splitting. Choose any map of $\mathbb{Z}$-modules $s \colon N \to M$ with $p \circ s = \text{id}_N$. It exists since we do *not* require that s is compatible with the G-action. Then a $\mathbb{Z}G$-splitting of $p \otimes_{\mathbb{Z}} \text{id}_{\mathbb{Z}G}$ is given by the $\mathbb{Z}G$-map $N \otimes_{\mathbb{Z}} \mathbb{Z}G \to M \otimes_{\mathbb{Z}} \mathbb{Z}G$ sending $n \otimes g$ to $gs(g^{-1}n) \otimes g$.

We will need a variant of tr that combines it with an induction map. This will yield additional control in the target category which is crucial for our argument. Let $\alpha \colon G \to F$ be a surjective group homomorphism, H be subgroup of finite index in F. Put $\overline{H} = \alpha^{-1}(H)$. We have defined induction and restriction in Section 2. Consider the functor

$$\text{tr}_\alpha := \text{tr}(\text{res}_\alpha \circ \text{ind}_H^F(-), -) \colon \text{mod}_{(\mathbb{Z},H)} \times \mathcal{O}^G(E, G, d_G) \to \mathcal{O}^G(E, G, d_G).$$

Define a quasi-metric $d_{G,H}$ on $G \times G/\overline{H}$ by

$$d_{G,\overline{H}}((g, a\overline{H}), (h, b\overline{H})) := \begin{cases} d_G(g, h) & \text{if } a\overline{H} = b\overline{H}, \\ \infty & \text{otherwise.} \end{cases}$$

The projection $p_H \colon G \times G/\overline{H} \to G$ induces a functor $P_H \colon \mathcal{O}^G(E, G \times G/\overline{H}, d_{G,\overline{H}}) \to \mathcal{O}(E, G, d_G)$ and we will see that we can lift tr_α against P_H. Define a functor

$$\widetilde{\text{tr}}_\alpha \colon \text{mod}_{(\mathbb{Z},H)} \times \mathcal{O}^G(E, G, d_G) \to \mathcal{O}^G(E, G \times G/\overline{H}, d_{G,\overline{H}})$$

as follows. For objects $T = (\mathbb{Z}^n, \rho) \in \text{mod}_{(\mathbb{Z},H)}$ and $M = (M_z)_{z \in G \times E \times [1,\infty)} \in \mathcal{O}^G(E, G, d_G)$ we define $\widetilde{\text{tr}}_\alpha(T, M)$ by setting

$$\big(\widetilde{\text{tr}}_\alpha(T, M)\big)_y := \mathbb{Z}^n \otimes_{\mathbb{Z}} M_z$$

for $y = (g, a\overline{H}, e, t) \in G{\times}G/\overline{H}{\times}E{\times}[1,\infty)$ and $z := (g,e,t)$. In order to write out $\widetilde{\mathrm{tr}}_\alpha$ for morphisms we need to choose representatives $g_0, \ldots, g_{m-1} \in G$ for $G/\overline{H} = \{g_0\overline{H}, \ldots, g_{m-1}\overline{H}\}$. For morphisms $f \in \mathtt{mod}_{(\mathbb{Z},H)}$ and $\psi = (\psi_{z,z'})_{z,z' \in G{\times}E{\times}[1,\infty)} \in \mathcal{O}^G(E, G, d_G)$ we define $\widetilde{\mathrm{tr}}_\alpha(f, \psi)$ by setting

$$\big(\widetilde{\mathrm{tr}}_\alpha(f,\psi)\big)_{y,y'} := \begin{cases} f \circ \rho(\alpha(g_k{}^{-1}g^{-1}g'g_{k'})) {\otimes}_\mathbb{Z} \psi_{z,z'} & \text{if } gg_k\overline{H} = g'g_{k'}\overline{H}, \\ 0 & \text{otherwise.} \end{cases}$$

for $y = (g, gg_k\overline{H}, e, t), y' = (g', g'g_{k'}\overline{H}, e', t') \in G{\times}G/\overline{H}{\times}E{\times}[1,\infty)$ and $z := (g, e, t), z' := (g', e', t')$. (The extra $G/\overline{H}$-factor incorporates the induction from H to F; the appearance of α incorporates the restriction along α.)

The following Lemma is a simple exercise in the definitions of tr_α and $\widetilde{\mathrm{tr}}_\alpha$.

Lemma 6.3.

    (i) $P_H \circ \widetilde{\mathrm{tr}}_\alpha$ and tr_α are equivalent functors.
    (ii) If ψ is an R-controlled morphism in $\mathcal{O}^G(E, G, d_G)$ and $f \in \mathtt{mod}_{(\mathbb{Z},H)}$ is any morphism, then $\widetilde{\mathrm{tr}}_\alpha(f, \psi)$ is R-controlled in $\mathcal{O}^G(E, G{\times}G/\overline{H}, d_{G,\overline{H}})$.

Proof. (i) To check this we unravel the definitions of tr_α and $\widetilde{\mathrm{tr}}_\alpha$ a bit. For $T = (\mathbb{Z}^n, \rho)$ we have

$$(\mathrm{res}_\alpha \circ \mathrm{ind}_H^F)(\mathbb{Z}^n, \rho) = (\mathbb{Z}^{nm}, \eta \circ \alpha) = \Big(\bigoplus_{k=0}^{m-1} \mathbb{Z}^n, \eta \circ \alpha \Big)$$

where η is as defined in the paragraph before Remark 2.2. It will be helpful to name each of the m copies of $\mathbb{Z}^n$, by $T_0, \ldots, T_{m-1}$. Then $\mathbb{Z}^{nm} = \bigoplus_{k=0}^{m-1} T_k$. Let $z = (g, e, t) \in G{\times}E{\times}[1,\infty)$. For $y = (g, gg_k\overline{H}, e, t)$ we have $\big(\widetilde{\mathrm{tr}}_\alpha(T, M)\big)_y = T_k {\otimes}_\mathbb{Z} M_z$ (as $T_k = \mathbb{Z}^n$). Therefore $\big(p_H \circ \widetilde{\mathrm{tr}}_\alpha(T, M)\big)_z \cong \bigoplus_{k=0}^{m-1} T_k {\otimes}_\mathbb{Z} M_z = \big(\mathrm{tr}_\alpha(T, M)\big)_z$. In particular, we have a canonical isomorphism $\tau_{T,M} : p_H \circ \widetilde{\mathrm{tr}}_\alpha(T, M) \cong \mathrm{tr}_\alpha(T, M)$.

We have to check that τ is natural with respect to morphisms (f, ψ). Inspection of the definition of η shows that for $\gamma \in G$ the (k, k')-block in $\eta \circ \alpha(\gamma)$ with respect to $\mathbb{Z}^{nm} = \bigoplus_{k=0}^{m-1} T_k$ is given by

$$\big(\eta \circ \alpha(\gamma)\big)_{k,k'} = \begin{cases} \rho(\alpha(g_k^{-1}\gamma g_{k'})) & \text{if } \gamma g_{k'}\overline{H} = g_k\overline{H}, \\ 0 & \text{otherwise.} \end{cases}$$

By definition

$$\big(\mathrm{tr}_\alpha(f, \psi)\big)_{z,z'} = (f \circ \eta(\alpha(g^{-1}g'))) {\otimes}_\mathbb{Z} \psi_{z,z'}$$

for $z = (g, e, t), z' = (g', e', t') \in G{\times}E{\times}[1,\infty)$. Thus with respect to the decomposition $\mathbb{Z}^{nm} = \bigoplus_{k=0}^{m-1} T_k$ the (k, k')-block of $\big(\mathrm{tr}_\alpha(f, \psi)\big)_{z,z'}$ is given by

$$\big(\big(\mathrm{tr}_\alpha(f, \psi)\big)_{z,z'}\big)_{k,k'} = \begin{cases} (f \circ \rho(\alpha(g_k^{-1}g^{-1}g'g_{k'}))) {\otimes}_\mathbb{Z} \psi_{z,z'} & \text{if } g'g_{k'}gg_k\overline{H} = \overline{H}, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this to the definition of $\widetilde{\mathrm{tr}}_\alpha$ we see that τ is natural for morphisms. (ii) By definition we have for $z = (g, gg_k\overline{H}, e, t)$, $z' = (g', g'g_{k'}\overline{H}, e', t')$

$$(\widetilde{\mathrm{tr}}_\alpha(f, \psi))_{z,z'} \neq 0 \quad \Longleftrightarrow \quad \big(gg_k\overline{H} = g'g_{k'}\overline{H} \text{ and } \psi_{z,z'} \neq 0\big)$$

where $z = (g, e, t)$, $z' = (g', e', t')$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

6.B. **Surjectivity of the P_k in** (4.1). In the remainder of this section we use the notation from Section 4. In particular G will from now on be a Farrell-Hsiang group. We denote by $(p_n)_* \colon \mathcal{O}^G(E, S_n, d_{S_n}) \to \mathcal{O}^G(E, G, d_G)$ the functor induced by the projection $p_n \colon S_n = G \times \coprod_{H \in \mathcal{H}_n} G/\overline{H} \to G$.

Proposition 6.4. *Let $n \in \mathbb{N}$.*

(i) *There are linear functors $F_n^+, F_n^- \colon \mathcal{O}^G(E, G, d_G) \to \mathcal{O}^G(E, S_n, d_{S_n})$ with the following two properties*
   - $K_*((p_n)_* \circ F_n^+) - K_*((p_n)_* \circ F_n^-)$ *is the identity on $K_*(\mathcal{O}^G(E, G, d_G))$;*
   - *if $R > 0$ and $\psi \in \mathcal{O}^G(E, G, d_G)$ is R-controlled, then $F_n^+(\psi)$ and $F_n^-(\psi)$ are both also R-controlled.*

(ii) *There are functors of additive categories with involutions $G_n^+ = (G_n^+, E_n^+)$ and $G_n^- = (G_n^-, E_n^-) \colon \mathcal{O}^G(E, G, d_G) \to \mathcal{O}^G(E, S_n, d_{S_n})$ with the following properties*
   - $L_*((p_n)_* \circ G_n^+) - L_*((p_n)_* \circ G_n^-)$ *is the identity on $L_*(\mathcal{O}^G(E, G, d_G))$;*
   - *if $R > 0$ and $\psi \in \mathcal{O}^G(E, G, d_G)$ is R-controlled, then $G_n^+(\psi)$ and $G_n^-(\psi)$ are both also R-controlled.*
   - *Denote by I both the involution on $\mathcal{O}^G(E, S_n, d_{S_n})$ and the involution on $\mathcal{O}^G(E, G, d_G)$. For each object $M \in \mathcal{O}^G(E, G, d_G)$, the isomorphisms $E_n^+(M) \colon G_n^+(I(M)) \to I(G_n^+(M))$ and $E_n^-(M) \colon G_n^-(I(M)) \to I(G_n^-(M))$ are 0-controlled.*

Proof. (i) By Theorem 5.1 (i) there are $\tau_H \in \mathrm{Sw}(\mathbb{Z}, H)$, $H \in \mathcal{H}_n$ such that $1_{\mathrm{Sw}} = \sum_{H \in \mathcal{H}_n} \mathrm{ind}_H^{F_n}(\tau_H) \in \mathrm{Sw}(\mathbb{Z}, F_n)$. Any element in $\mathrm{Sw}(\mathbb{Z}, H)$ can be written as the difference of the classes of two modules. Pick modules T_H^+ and $T_H^- \in \mathrm{mod}_{(\mathbb{Z}, G)}$, $H \in \mathcal{H}_n$ such that $\tau_H = [T_H^+] - [T_H^-]$. Because res_{α_n} sends $1_{\mathrm{Sw}} \in \mathrm{Sw}(\mathbb{Z}, F_n)$ to $1_{\mathrm{Sw}} \in \mathrm{Sw}(\mathbb{Z}, G)$ we obtain

$$1_{\mathrm{Sw}} = \sum_{H \in \mathcal{H}_n} [\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^+)] - [\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^-)] \in \mathrm{Sw}(\mathbb{Z}, G)$$

For $H \in \mathcal{H}_n$ we have a canonical inclusion $G \times G/\overline{H} \to S_n = G \times \coprod_{K \in \mathcal{H}_n} G/\overline{K}$ that induces an inclusion $\mathcal{O}^G(E, G \times G/\overline{H}, d_{G, \overline{H}}) \to \mathcal{O}^G(E, S_n, d_{S_n})$. Define $F_H^\pm$ as the composition of $\widetilde{\mathrm{tr}}_{\alpha_n}(T_H^\pm, -)$ with this inclusion. Then $K_*((p_n)_* \circ F^\pm H) = K_*(\mathrm{tr}(\mathrm{res}_{\alpha_n} \mathrm{ind}_H^{F_n}(T_H^\pm), -))$ by Lemma 6.3 (i). Define now

$$F_n^\pm := \bigoplus_{H \in \mathcal{H}_n} F_H^\pm.$$

The functor tr is exact by Lemma 6.1 and so Proposition 5.3 applies. Therefore we can compute for all $a \in K_*(\mathcal{O}^G(E, G, d_G))$

$$K_*((p_n)_* \circ F_n^+)(a) - K_*((p_n)_* \circ F_n^-)(a)$$
$$= \sum_{H \in \mathcal{H}_n} K_*((p_n)_* \circ F_H^+)(a) - K_*((p_n)_* \circ F_H^-)(a)$$
$$= \sum_{H \in \mathcal{H}_n} K_*(\mathrm{tr}(\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^+)))(a) - K_*(\mathrm{tr}(\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^-)))(a)$$
$$= \sum_{H \in \mathcal{H}_n} \mu_{\mathrm{tr}}([\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^+)] \otimes a) - \mu_{\mathrm{tr}}([\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^-)] \otimes a)$$
$$= \mu_{\mathrm{tr}}((\sum_{H \in \mathcal{H}_n} [\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^+)] - [\mathrm{res}_{\alpha_n} \circ \mathrm{ind}_H^{F_n}(T_H^-)]) \otimes a)$$
$$= \mu_{\mathrm{tr}}(1_{\mathrm{Sw}} \otimes a) = a$$

If $R > 0$ and $\psi \in \mathcal{O}^G(E, G, d_G)$ is R-controlled then each $F_H^\pm(\psi)$ is R-controlled, because of the control property of $\widetilde{\mathrm{tr}}_{\alpha_n}$ (Lemma 6.3 (ii)) and because $G \times G/\overline{H} \to S_n$

is an isometric embedding. The direct sum of R-controlled morphisms is again R-controlled and therefore $F^+(\psi)$ and $F^-(\psi)$ are both R-controlled.

(ii) We can proceed exactly as in the K-theory case. By Theorem 5.1 (ii) there are $\sigma_H \in \mathrm{GW}(\mathbb{Z}, H)$, $H \in \mathcal{H}_n$ such that $1_{\mathrm{GW}} = \sum_{H \in \mathcal{H}_n} \mathrm{ind}_{F_n}^H(\sigma_H)$. Any element in $\mathrm{GW}(\mathbb{Z}, H)$ can be written as the difference of the classes of two symmetric forms. Pick symmetric forms (T_H^+, φ_H^+) and (T_H^-, φ_H^-) over $\mathrm{mod}_{(\mathbb{Z}, G)}$, $H \in \mathcal{H}_n$ such that $\sigma_H = [(T_H^+, \varphi_H^+)] - [(T_H^-, \varphi_H^-)]$. Define $G_H^{\pm}$ as the composition of $\mathrm{tr}_{\alpha_n}((T_H^+, \varphi_H^+), -)$ with the inclusion $\iota_H \colon \mathcal{O}^G(E, G \times G/\overline{H}, d_{G, \overline{H}}) \to \mathcal{O}^G(E, S_n, d_{S_n})$ and set

$$G_n^{\pm} := \bigoplus_{H \in \mathcal{H}_n} G_H^{\pm}.$$

As in the K-theory case it follows (using now Proposition 5.4) that for all $b \in L_*(\mathcal{O}^G(E, G, d_G))$ we have

$$L_*((p_n)_* \circ G_n^+)(b) - L_*((p_n)_* \circ G_n^-)(b) = b$$

and that $G_n^{\pm}(\psi)$ is R-controlled, whenever ψ is R-controlled.

It remains to prove the final claim. Let M be an object from $\mathcal{O}^G(E, G, d_G)$. Then

$$\begin{aligned}
G_n^{\pm}(I(M)) = \bigoplus_H G_H^+(I(M)) &= \bigoplus_H \iota_H(\widetilde{\mathrm{tr}}_{\alpha_n}(T_H^{\pm}, I(M)) \\
I(G_n^{\pm}(M)) = \bigoplus_H I(G_H^+(M)) &= \bigoplus_H I(\iota_H(\widetilde{\mathrm{tr}}_{\alpha_n}(T_H^{\pm}, M) \\
&= \bigoplus_H \iota_H(\widetilde{\mathrm{tr}}_{\alpha_n}((T_H^{\pm})^*, I(M))
\end{aligned}$$

and

$$E_n^{\pm}(M) = \bigoplus_H \iota_H(\widetilde{\mathrm{tr}}_{\alpha_n}(\varphi_H^{\pm}, \mathrm{id}_{I(M)}).$$

The control claim follows from Lemma 6.3 (ii) because $\mathrm{id}_{I(M)}$ is 0-controlled. $\quad\square$

Theorem 6.5.

  (i) *For all $a \in K_*(\mathcal{O}^G(E, G, d_G))$ there is $\hat{a} \in K_*(\mathcal{O}^G(E, (S_n, d_{S_n})_{n \in \mathbb{N}})$ such that for all k we have $(K_*(P_k))(\hat{a}) = a$.*

  (ii) *For all $b \in L_*(\mathcal{O}^G(E, G, d_G))$ there is $\hat{b} \in L_*(\mathcal{O}^G(E, (S_n, d_{S_n})_{n \in \mathbb{N}})$ such that for all k we have $(L_*(P_k))(\hat{b}) = b$.*

Proof. (i) Let F_n^+, F_n^- be the sequences of functors from Proposition 6.4 (i). Because of the control property in 6.4 (i) the product functors

$$\prod_n F_n^+ \colon \mathcal{O}^G(E, G, d_G) \to \prod_n \mathcal{O}^G(E, S_n, d_{S_n})$$

lift uniquely to functors

$$F^{\pm} \colon \mathcal{O}^G(E, G, d_G) \to \mathcal{O}^G(E, (S_n, d_{S_n})_{n \in \mathbb{N}}).$$

Then $P_k \circ F^{\pm} = (p_k)_* \circ F_k^{\pm}$ for all $k \in \mathbb{N}$. Thus the first assertion in 6.4 (i) implies that $K_*(P_k)(K_*(F^+)(a) - K_*(F^-)(a)) = a$ for all $a \in \mathcal{O}^G(E, G, d_G)$. Therefore we can set $\hat{a} := K_*(F^+)(a) - K_*(F^-)(a)$.

(ii) For L-theory we can argue exactly as we did for K-theory, now using the $G_n^{\pm}$ from Proposition 6.4 (ii). Here the third assertion in 6.4 (ii) is needed to ensure that the $E_n^{\pm}$ can be combined to a natural transformation, just as the second assertion is needed to ensure that the $G_n^{\pm}$ can be combined to a functor. $\quad\square$

7. The functor F

We use the notation from Section 4. Note first that for any subgroup U of G there is a bijection of G-sets $G \times G/U \to \operatorname{ind}_G^U \operatorname{res}_U^G G = G \times_U G$ defined by $(a, gU) \mapsto (g, g^{-1}a)$; the inverse is given by $(g, b) \mapsto gb, gU$. (We use the diagonal G-action on $G \times G/U$.) For $H \in \mathcal{H}_n$ we obtain a G-map $\tilde{f}_H \colon G \times G/\overline{H} \to \operatorname{ind}_G^{\overline{H}} E_H$ by composing this bijection (for $U = \overline{H}$) with $\operatorname{ind}_G^{\overline{H}} f_H \colon \operatorname{ind}_G^{\overline{H}} G \to \operatorname{ind}_G^{\overline{H}} E_H$. Define the G-map $f_n \colon S_n \to X_n$ by

$$f_n(a, g\overline{H}) := (a, \tilde{f}_H(a, g\overline{H}) = (a, g, f_H(g^{-1}a)),$$

for $a, g \in G$ and $H \in \mathcal{H}_n$.

Proposition 7.1. *The sequence of maps $(f_n)_{n \in \mathbb{N}}$ induces a functor*

$$F \colon \mathcal{O}^G(E, (S_n, d_{S_n})_{n \in \mathbb{N}}) \to \mathcal{O}^G(E, (X_n, d_{X_n})_{n \in \mathbb{N}}).$$

For all k we have $q_k \circ F = p_k$.

Proof. We need to show that the sequence $(f_n)_{n \in \mathbb{N}}$ is compatible with the metric control conditions for the sequences of quasi-metrics $(d_{S_n})_{n \in \mathbb{N}}$ and $(d_{X_n})_{n \in \mathbb{N}}$ more precisely we need to show that for any $r \in (0, \infty)$ there is $R \in (0, \infty)$ such that for all n and $s, s' \in S_n$ the implication

$$(7.2) \qquad d_{S_n}(s, s') < r \quad \implies \quad d_{X_n}(f_n(s), f_n(s')) < R$$

holds.

Let $r \in (0, \infty)$ be given. The G-action on S_n is cofinite, the quasi-metrics d_{S_n} and d_{G_n} are G-invariant and f_n is G-equivariant. For each $s \in S_n$ there are only finitely many $s' \in S_n$ such that $d_{S_n}(s', s) < r$, because the word metric d_G has this property on G. This implies that $D_r := \{d_{X_n}(f_n(s), f_n(s')) \mid n < r, s, s' \in S_n, d_{S_n}(s, s') < r\}$ is a finite set. We can therefore define $R := 1 + r + \max D_r$. We claim that then (7.2) holds for all n and all $s, s' \in S_n$. If $n < r$, then this is clear from the definition of R. Let $n > r$ and $s, s' \in S_n$ with $d_{S_n}(s, s') < r$. Write $s = (a, g\overline{H})$ and $s' = (a', g'\overline{H'})$ with $H, H' \in \mathcal{H}_n, a, a', g, g' \in G$. Since $d_{S_n}(s, s') < r < \infty$ it follows from the definition of d_{S_n} that $H = H'$, $g\overline{H} = g'\overline{H'}$ and $d_G(a, a') < r < n$. Since d_G is G-invariant we also have $d_G(g^{-1}a, g^{-1}a') < n$. We conclude from the crucial contracting property of f_H that $d_{E_H}^1(f_H(g^{-1}a), f_H(g^{-1}a')) < \frac{1}{n}$. Since $s = (a, g\overline{H})$, $s' = (a', g\overline{H})$ we have $f_n(s) = (a, g, f_H(g^{-1}a))$, $f_n(s') = (a', g, f_H(g^{-1}a'))$. Thus

$$
\begin{aligned}
d_{X_n}(f_n(s), f_n(s')) &= d_{X_n}((a, g, f_H(g^{-1}a)), (a', g, f_H(g^{-1}a'))) \\
&= d_G(a, a') + n \cdot d_{\operatorname{ind}_G^{\overline{H}} E_H}^1((g, f_H(g^{-1}a)), (g, f_H(g^{-1}a'))) \\
&= d_G(a, a') + n \cdot d_{E_H}^1(f_H(g^{-1}a), f_H(g^{-1}a')) \\
&< r + n \cdot \frac{1}{n} = r + 1 < R.
\end{aligned}
$$

This proves our claim. Thus $(f_n)_{n \in \mathbb{N}}$ induces a functor F.

For the canonical projections $p_k \colon S_k \to G$ and $q_k \colon X_k \to G$ we have $q_k \circ f_k = p_k$. This implies that $Q_k \circ F = P_k$. $\qquad\qquad\square$

Appendix A. Applications and examples of Farrell-Hsiang groups

Proofs of the Farrell-Jones Conjecture often combine methods from controlled topology (for example our Theorem 1.2) with group theoretic and geometric considerations (for example to show that certain groups are Farrell-Hsiang groups with respect to some family $\mathcal{F}$) and an induction using the transitivity principle [12, Theorem A.10]. The transitivity principle asserts that for families of groups $\mathcal{F} \subseteq \mathcal{G}$ the Farrell-Jones Conjecture for G holds relative to $\mathcal{F}$ provided a) the Farrell-Jones

Conjecture for G holds relative to $\mathcal{G}$ and b) for any $H \in \mathcal{G}$ the Farrell-Jones Conjecture holds relative to $\mathcal{F}$. In the following we briefly discuss some results from [1] and their connection to Farrell-Hsiang groups.

Many crystallographic groups are Farrell-Hsiang groups relative to interesting families of subgroups, see [1, Proofs of Lemma 2.8, Lemma 2.15, Theorem 2.1]. For example $\mathbb{Z}^2 \rtimes_{-\mathrm{id}} \mathbb{Z}/2$ is a Farrell-Hsiang group relative to VCyc. In combination with the transitivity principle this yields a proof of the Farrell-Jones Conjecture with additive categories as coefficients for virtually finitely generated abelian groups. This generalizes [17] where only untwisted ring as coefficients are treated. (The version with additive categories as coefficients has better inheritance and transitivity properties and encompasses the so called fibered version).

The main motivation for this paper is that its methods apply to situations, where the known techniques for virtually abelian groups do not work anymore. Namely, special affine groups are Farrell-Hsiang groups relative to the family of virtually finitely generated abelian groups, see [1, Proof of Proposition 3.40]. This fact is a key ingredient for the proof of the Farrell-Jones Conjecture with additive categories as coefficients for virtually poly-cyclic groups and finally for cocompact lattices in virtually connected Lie groups in [1].

In summary, our axiomatic treatment of the Farrell-Hsiang method in Theorem 1.2 encapsulates completely the input of controlled topology to [1], separates it from the necessary group theoretic and geometric arguments carried out there, and applies for instance to special affine groups.

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